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Analytic Representation of Envelope Surfaces Generated by Motion of Surfaces of Revolution

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Abstract—A modified DG/K (Differential Geometry/Kinematics) approach to analytical solution of envelope surfaces generated by continuous motion of a generating surface – general surface of revolution – is presented in this paper. This approach is based on graphical representation of an envelope surface in parametric space of a solid generated by continuous motion of the generating surface. Based on graphical analysis, it is possible to decide whether the envelope surface exists and recognize the expected form of unknown analytical representation of the envelope surface. The obtained results can be used in application of envelope surfaces in mechanical engineering, especially in 3-axis and 5-axis point and flank milling of freeform surfaces.

Index Terms—Characteristic curve, Eenvelope surface, Explicit surface, Flank milling, Parametric curve, Parametric surface, Surface of revolution, Tangent plane, Tangent vector.

1 Introduction

DETERMINATION of analytic representation of an envelope surface generated by continuous motion of a generating surface in sufficiently general conceptionof the generating surface as well as the trajectory of motion is a very challenging problem of kinematic geometry and its applications in mechanical engineering [1], [2]. From application point of view, the generating surface corresponds to the milling tool in CNC (Computer Numeric Control) milling, the trajectory of motion corresponds to the milling path and the resulting envelope surface corresponds to the machined surface.

Academic textbooks describe primarily synthetical solution of envelope surface problem based on pointwise construction of characteristic curve of envelope surface, i.e. the curve along which the generating and envelope surfaces are touching at each instant during the whole motion. If an analytical solution is published, it is either restricted to implicit expression of the figures or such a motion (linear motion, rotation, helical motion) and such a generating surface (plane [3] or spherical surface [4], [5]) are chosen to obtain envelope surface its analytical representation is known in advance of solution. So called DG/K (Differential Geometry/Kinematics) approach to find analytical representation of envelope surface in sufficiently general conception can be find in [2], [6]. In DG/K approach, the envelope surface is considered to be the surface which is touching every surface of the univariate family of generating surfaces along a characteristic curve. Then, the envelope surface is represented by a set of characteristic curves.

In this paper, a modified DG/K approach is described. The envelope surface is considered to be the surface in the solid generated by univariate motion of the generating surface at whose every pointall three tangent vectors to all three parametric curves of this solid are coplanar. The main contribution of this approach consists in implementation of a graphical analysis of the problem in parametric space of the generated solid. Based on graphical analysis, it is possible to decide whether the envelope surface exists or not. If the envelope surface exists, it is possible to recognize expected form of analytical representation of the envelope surface and determine its concrete expression.

The paper is organized as follows. The specification of motion is given in Section 2. Section 3 deals with the definition and basic properties of an envelope surface generated by univariate motion of a generating surface. The method developed for determination of analytical representation of the envelope surface is described in Section 4. Examples of envelope surfaces generated by motion of general surface of revolution are given in Section 5.

2 Motion

Envelope surface is a figure generated by motion of another surface. To be able to investigate the generated figure, it is necessary to specify the motion, first. Let $\left(\Omega(t), \xi(t), \eta(t), \zeta(t)\right)$ be Cartesian local coordinate system of moving figure. Analytic representation of motion along a trajectory $\mathbf{T}(t)$ in homogenous coordinates of projective extension of Euclidean three-dimensional space (3D space) given by

$$\mathbf{T}(t) = (x(t), y(t), z(t), 1),\ t \in [t_1, t_2] \tag{1}$$

is transformation matrix

$$\mathbf{G}(t) = \begin{pmatrix} \xi_x(t) & \xi_y(t) & \xi_z(t) & 0 \\ \eta_x(t) & \eta_y(t) & \eta_z(t) & 0 \\ \zeta_x(t) & \zeta_y(t) & \zeta_z(t) & 0 \\ x(t) & y(t) & z(t) & 1 \end{pmatrix}, t \in [t_1, t_2], \tag{2}$$

where $\xi_x(t)$, $\xi_y(t)$, $\xi_z(t)$ and $\eta_x(t)$, $\eta_y(t)$,$\eta_z(t)$ and $\zeta_x(t)$, $\zeta_y(t)$, $\zeta_z(t)$ are cosine of angles formed by axes $\xi(t)$,$\eta(t)$ and $\zeta(t)$ of local coordinate system of moving figure and axes of world coordinate system x, y and z in the given order (provided that $\xi(t)$, $\eta(t)$ and $\zeta(t)$ are univariate real functions of real variable t, defined, continuous and at least once differentiable on $[t_1, t_2]$).

If the trajectory is a freeform curve, the motion is called curvilinear. An example of curvilinear motion is motion of

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milling tool in CNC 5-axis milling. Two types of motion can be distinguished: general motion when the mutual position of the local coordinate system of moving figure with respect to the natural coordinate system of trajectory (given by tangent, normal and binormal line of the trajectory) is preserved during the whole motion and translational motion when the mutual position of the local coordinate system of moving figure with respect to the world coordinate system is preserved during the whole motion.

3ENVELOE SURFACE

To derive generally analytical representation of an envelope surface generated by motion of a generating surface, let us consider the generating surface in 3D space with analytic representation

$$\mathbf{S}(u,v) = (x(u,v), y(u,v), z(u,v), 1),$$
$$u \in [u_1, u_2],\ v \in [v_1, v_2], \tag{3}$$

and transformation matrix (2) of continuous motion. Then a solid

$$\mathbf{B}(u,v,t) = \mathbf{S}(u,v) \cdot \mathbf{G}(t),$$
$$u \in [u_1, u_2],\ v \in [v_1, v_2],\ t \in [t_1, t_2] \tag{4}$$

is generated by motion of the generating surface (3) along the trajectory (1). A part

$$\mathbf{E}(s,t),\ s \in [s_1, s_2],\ t \in [t_1, t_2] \tag{5}$$

of superficies of this solid is called envelope surface if the following conditions are satisfied.

1. Surface (5) and any uv-parametric surface
$$\mathbf{B}(u,v,\gamma_i), \gamma_i \in [t_1, t_2], i = 0,1,\cdots,n \tag{6}$$
of solid (4) are tangent along s-parametric curve $\mathbf{E}(s,\gamma_i)$ of surface (5) called characteristic curve of envelope surface.
2. At each point of surface (5) exists common tangent plane and normal line of surface (5) and one uv-parametric surface (6) of solid (4).
3. A surface which is simultaneously a part of surface (5) and any uv-parametric surface (6) does not exist.

As the characteristic curve is located on the generating surface, each point of characteristic curve moves together with the generating surface. To satisfy the contact of the generating and envelope surfaces along the characteristic curve, the common tangent plane of the generating and envelope surface has to exist. The tangent plane at any point of the generating surface is given by tangent vectors to the parametric u- and v-curves of the generating surface. Considering the motion of the generating surface along the trajectory, it is necessary to exclude points of the generating surface located inside the solid (4). To exclude these points, the tangent plane has to contain the tangent vector of parametric t-curve of the solid, too. Obviously, this condition is fulfilled in case the tangent vectors of all three parametric curves of solid (4) are coplanar, i.e. their mixed product is equal to zero.

Tangent vectors of parametric curves of solid (4) are given by first partial derivatives of cordinate functions of the solid (4) with respect to individual varialbes, i.e.

$$\mathbf{B}^w(u,v,t) = \frac{\partial \mathbf{B}(u,v,t)}{\partial w} = (x^w_{\mathbf{B}}(u,v,t), y^w_{\mathbf{B}}(u,v,t), z^w_{\mathbf{B}}(u,v,t), 0),$$
$$w = u, v, t. \tag{7}$$

The mixed product

$$\mathbf{B}^u(u,v,t) \cdot [\mathbf{B}^v(u,v,t) \times \mathbf{B}^t(u,v,t)]$$

of tangent vectors $\mathbf{B}^u(u,v,t)$, $\mathbf{B}^v(u,v,t)$ and $\mathbf{B}^t(u,v,t)$ can be calculated as the determinant $D(u,v,t)$ of matrixhaving the three vectors as its rows (or columns), i.e.

$$D(u,v,t) = \begin{vmatrix} x^u_B(u,v,t) & y^u_B(u,v,t) & z^u_B(u,v,t) \\ x^v_B(u,v,t) & y^v_B(u,v,t) & z^v_B(u,v,t) \\ x^t_B(u,v,t) & y^t_B(u,v,t) & z^t_B(u,v,t) \end{vmatrix}. \tag{8}$$

If there exists a solution of equation

$$D(u,v,t) = 0, \tag{9}$$

then this solution is analytical representation of unknown envelope surface in parametric space (u,v,t) of the solid $\mathbf{B}(u,v,t)$ generated by motion of the generating surface.

Example of the whole procedure of envelope surface generation is given in fig. 1. The generating surface is a general surface of revolution $\mathbf{S}(u,v)$ given by axis of revolution o and meridian m, see fig. 1 (a). In the case of general surface of revolution the meridian is a freeform curve. To simplify the picture, the trajectory of motion is planar freeform curve $\mathbf{T}(t)$ located in the plane perpendicular to the axis o. Several positions of the generating surface along the trajectory is drawn in fig. 1 (b). The generating surface moves by general motion because its position with respect to the trajectory is preserved. In particular, the meridian plane of the generating surface and the normal plane of the trajectory are identical. The solid $\mathbf{B}(u,v,t)$ generated by motion of generating surface along the trajectory is drawn in fig. 1 (c). The resulting envelope surface $\mathbf{E}(s,t)$ together with characteristic curve $\mathbf{C}(s)$ is shown in fig. 1 (d). Note that in practical applications, this type of motion and envelope surface generation corresponds to 2.5-axis flank milling.

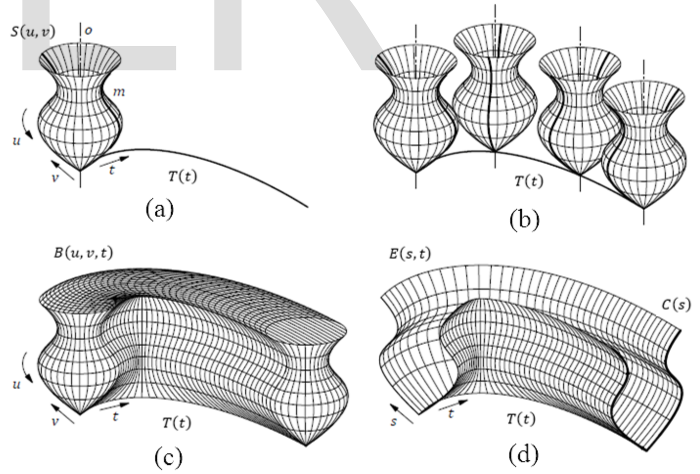

Fig. 1: Process of envelope surface generation

4 ANALYTIC REPRESENTATION OF ENVELOPE SURFACE

There are two basic situations when solving envelope surface problem: the shape and analytical representation of envelope surface is known in advance, and for the given generating surface and the given motion we know neither wheatear the envelope surface exists nor its possible shape and analytical representation. Envelope surfaces generated by motion of a sphere belong to the first category [8], see examples in fig. 2.

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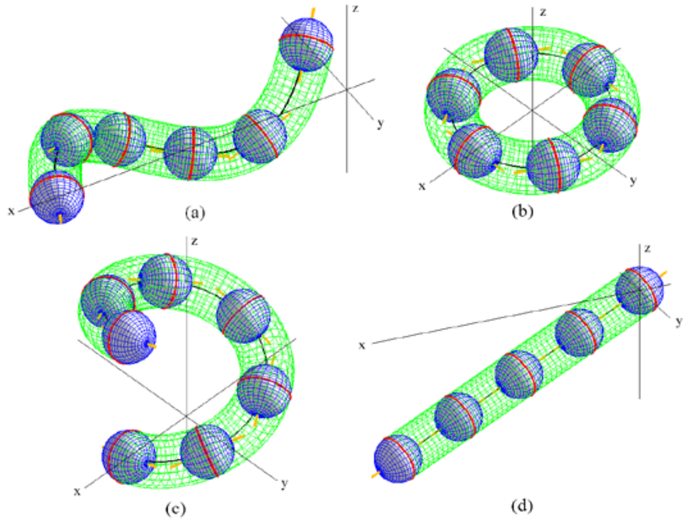

Fig. 2: Example of envelope surfaces generated by motion of a sphere (a) canal surface (generated by general motion of a sphere), (b) torus (generated by rotary motion of a sphere), (c) serpentine of Archimedes (generated by helical motion of a sphere), (d) cylinder of revolution (generated by linear motion of a sphere)

If the shape and analytical representation of envelope surface is unknown, it is possible to find it by means of the method which has been developed to solve envelope surfaces problem in the most general conception. This method is based on graphical analysis of the problem in parametric space (u, v, t) of the solid (4). Individual steps of the method are as follows.

1. **Solid**$\mathbf{B}(u, v, t)$ – Firstly, the analytical representation of the solid $\mathbf{B}(u, v, t)$ acc. (4) and its tangent vectors $\mathbf{B}^u(u, v, t)$, $\mathbf{B}^v(u, v, t)$ and $\mathbf{B}^t(u, v, t)$ acc. (7) is necessary to determine.
2. **Determinant** $D(u, v, t)$ – Next, the determinant $D(u, v, t)$ acc. (8) is expressed. Generally, the determinant is a function of three variables. To be able to realize graphical representation of equation $D(u, v, t) = 0$ is therefore suitable to choose sufficient number of instants $t = \gamma_i,\ i = 0,1, \cdots, n$ (the function of three variables becomes an explicit function of two variables) and draw in three-dimensional graph both the explicit function$D(u, v, \gamma_i)$ and function $z = 0$. Thus, n three-dimensional graphs is obtained.
3. **Animation** – All three-dimensional graphs are arranged into animation sequence.
4. **Existence of solution** – If there exists an intersection curve of the surface $D(u, v, \gamma_i)$ and $z = 0$ for each $i = 0,1, \cdots, n$, the envelope surface exists. The intersection curve is a representation of characteristic curve of envelope surface in parametric space (u, v, t) of the solid (4).
5. **Non-variable or variable shape of characteristic curve** – If the intersection curve has identical shape during the whole animation, solution of equation $D(u, v, t) = 0$ does not depend on t and, consequently, characteristic curve of envelope surface has non-variable shape. If the shape of intersection curve is not constant during animation, the solution of equation $D(u, v, t) = 0$ depends on t and, therefore, the characteristic curve of envelope surface has variable shape.
6. **Type of solution** – Based on the shape of intersection curve it is possible to distinguish the following type of solution: envelope surface is vt- or ut-parametric surface of the solid $\mathbf{B}(u, v, t)$, envelope surface is an explicit surface of the solid and envelope surface is a surface in the solid expressed parametrically, see Sections 4.1, 4.2 and 4.3.

4. 1 Envelope Surface Is Parametric Surface of the Solid

Envelope surface is vt- or ut-parametric surface of the solid $\mathbf{B}(u, v, t)$ in case the intersection curve is straight line
$u = \alpha, \alpha \in [u_1, u_2]orv = \beta, \beta \in [v_1, v_2]$
of non-variable shape during the whole animation. Then it is possible to express the analytic representation of envelope surface as
$\mathbf{E}(v, t) = \mathbf{B}(\alpha, v, t)$or $\mathbf{E}(u, t) = \mathbf{B}(u, \beta, t)$.

The characteristic curve of envelope surface is v- or u-parametric curve of the solid $\mathbf{B}(u, v, t)$ in this case. Envelope surfaces of this type are generated by motion of a sphere (fig. 2), general motion of a surface of revolution along a planar trajectory (fig. 1) and by rotary motion of a surface of revolution when axis of rotary motion and axis of the generating surface are parallel (see example in Section 5.1) or intersecting.

4. 2 Envelope SurfaceIsExplicitSurface in the Solid

Envelope surface is an explicit surface in the solid $\mathbf{B}(u, v, t)$. This situation occurs in case the intersection curve represents a graph of univariate functionfor each $\gamma_i, i = 0,1, \cdots, n$, i.e.

$u = u(v)$or$v = v(u)$ (10)

in the case of non-variable shape and

$u = u(v, t)$or$v = v(u, t)$ (11)

in the case of variable shape of the intersection curve.

Using (10), the analytical representation of the envelope surface is given by

$$\mathbf{E}(v, t) = \mathbf{B}(u(v), v, t)\text{or}\mathbf{E}(u, t) = \mathbf{B}(u, v(u), t) \tag{12}$$

and using (11), the analytical representation of the envelope surface is given by

$$\mathbf{E}(v, t) = \mathbf{B}(u(v, t), v, t)\text{or } \mathbf{E}(u, t) = \mathbf{B}(u, v(u, t), t). \tag{13}$$

Envelope surfaces (12) are generated by helical and linear motion of a surface of revolution and by rotary motion of a surface of revolution when the axis of rotary motion and the axis of the generated surface are skew lines (see example in Section 5.2). Envelope surfaces (13) are generated by general or generalized translational motion of surface of revolution along a spatial trajectory (see example in Section 5.3).

Note that if the shape of intersection curve is so complicated that it cannot be described by one equation, it is possible to find corresponding function (10) or (11) in piecewise form or use graphical solution described in Section 4.3.

4. 3 Envelope SurfaceIsaSurface in the Solid Expressed Parametrically

Envelope surface is a surface in the solid $\mathbf{B}(u, v, t)$ expressed parametrically. This situation occurs if there does not exist a solution of equation $D(u, v, t) = 0$ in previously mentioned forms or due to the complexity of intersecting curve it is impossible to find it. Then it is useful to express the solution parametrically in the following form

$$u = u(s),\ v = v(s) \tag{14}$$

in case of non-variable shape and

$$u = u(s, t),\ v = v(s, t) \tag{15}$$

in case of variable shape of the intersection curve.

Using (14), the analytical representation of the envelope surface is given by

$$\mathbf{E}(s,t) = \mathbf{B}(u(s), v(s), t) \qquad (16)$$

and using (15), the analytical representation of the envelope surface is given by

$$\mathbf{E}(s,t) = \mathbf{B}(u(s,t), v(s,t), t). \qquad (17)$$

Depending on the complexity of the shape of generating surface, envelope surfaces (16) and (17) can be generated by any motion.

Note that in the most general approach it is possible to use planar two-dimensional graph of intersection curve drawn in (u,v) plane and read parametric coordinates u and v of sufficient number of points along the intersection curve. These points can be consequently fitted by interpolation or approximationcurve to obtain mathematical model of (14) or (15).

5 Examples

In this section, examples of envelope surfaces generated by motion of general surface of revolution drawn in fig. 3 are given. The generating surface is depicted in basic position, i.e. its local coordinate system $(\Omega, \xi, \eta, \zeta)$ is identical with the world coordinate system $(0, x, y, z)$. By rotary motion of free-form left principle half-meridian given by

$$\mathbf{M}(v) = (x(v), y(v), z(v), 1) =$$
$$= (20v^3 - 33v^2 + 15v, 0{,}9v^3 - 15v^2 + 12v, 1), v \in [0,1] \qquad (18)$$

along$z$-axis, the analytic representation of surface of revolution is obtained

$$\mathbf{S}(u,v) = (x(u,v), y(u,v), z(u,v), 1), u \in [0,2\pi], v \in [0,1], \qquad (19)$$
$$x(u,v) = (20v^3 - 33v^2 + 15v)\cos u\,,$$
$$y(u,v) = (20v^3 - 33v^2 + 15v)\sin u,$$
$$z(u,v) = 9v^3 - 15v^2 + 12v.$$

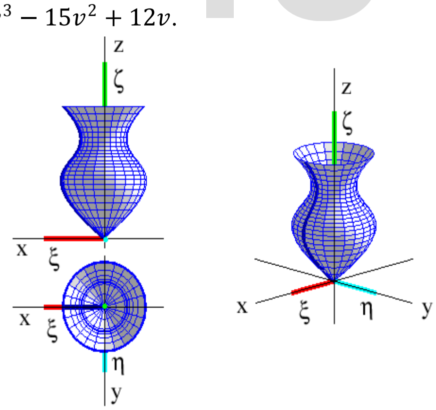

Fig. 3: General surface of revolution in basic position (top, front and axonometric view)

In the following sections the transformation matrices for all the considered types of motionare given. Analytical representation of the generated solid (4) is too largefor all the presented examples, therefore, it is not given here.

5.1Envelope Surface Is Parametric Surface of the Solid

In this section an example of envelope surface generated by rotary motion of surface of revolution (19) is shown. The axis of rotary motion and axis of surface of revolution are parallel. The trajectory of motion is a circle of radius r lying in (x,y) plane given by

$$\mathbf{T}(t) = (r\cos t, r\sin t, 0\,, 1), t \in [0,2\pi]. \qquad (20)$$

The transformation matrix (2) of rotary motion in this case is

$$\mathbf{G}(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r\cos t & r\sin t & 0 & 1 \end{pmatrix}. \qquad (21)$$

The determinant (8) has the following form

$$D(u,v) = rx(v)z'(v)\sin u,$$

see fig. 4, where the spatial graph of this surface is drawn. The determinant is independent on t, therefore the characteristic curve has non-variable shape – equal to the meridian (18).

Solution of equation $D(u,v) = 0$ is

$$u = 0, u = \pi, \qquad (22)$$

i.e. the envelope surface is two-branchesvt-parametric surface of the generated solid. It is obvious from graphical representation of (22), i.e. from the mapping of envelope surface in parametric space (u,v,t) of the generated solid shown in fig. 5. As each branche of the envelope surface is mapped into a plane parallel with (u,v) plane, it is a parametric surface of the solid.

Two branches of envelope surface in 3D space are given by

$$\mathbf{E}(v,t) = (\cos t[x(v) + r], \sin t[x(v) + r]\,, z(v), 1),$$
$$\mathbf{E}^*(v,t) = (\cos t[x(v) + r], \sin t[x(v) + r]\,, z(v), 1),$$

see fig. 6.

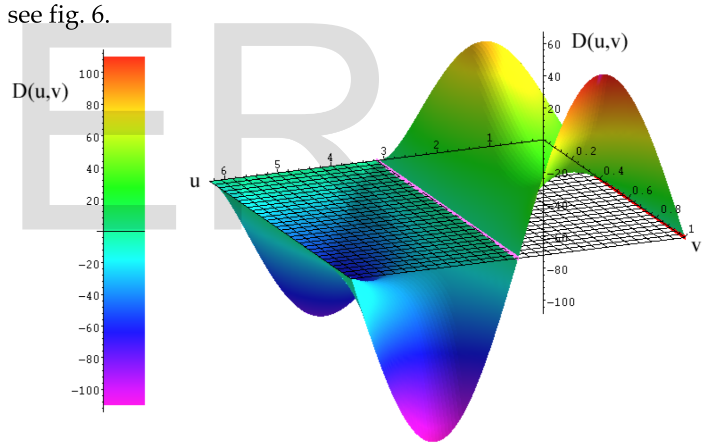

Fig. 4: Graphical solution of equation $D(u,v) = 0$ for envelope surface generated by rotary motion of general surface of revolution when axis of rotary motion and axis of the generating surface are parallel

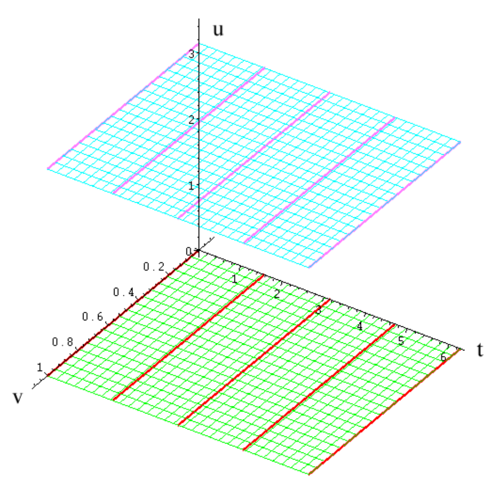

Fig. 5: Envelope surface (with several characteristic curves) generated by

rotary motion of general surface of revolution when axis of rotary motion and axis of the generating surface are parallel (mapping in parametric space (u, v, t) of the generated solid)

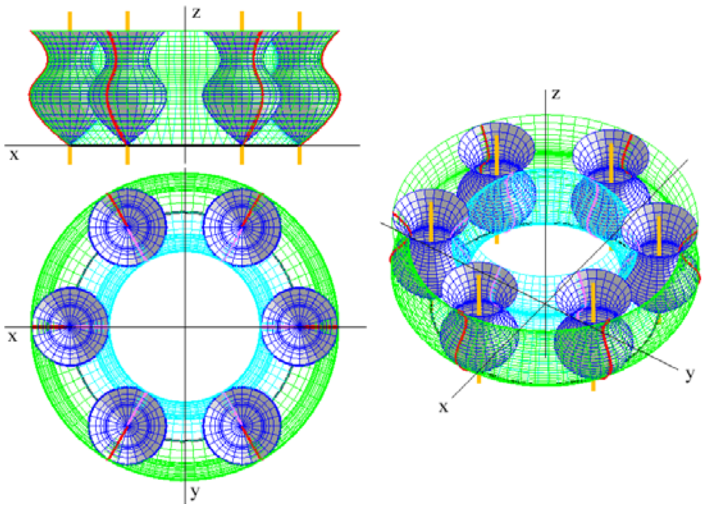

Fig. 6: Envelope surface generated by rotary motion of general surface of revolution when axis of rotary motion and axis of the generating surface are parallel (top, front and axonometric view in 3D space)

5.2 Envelope Surface Is ExplicitSurface of the Solid, Characterictic Curve Has Non-Variable Shape

Here an example of envelope surface generated by rotary motion of surface of revolution (19) when axis of rotary motion and axis of the surface of revolution are skew. The trajectory of motion is the circle given by (20), transformation matrix of rotary motion is given by (21).

The difference from the previous example is the initial position of surface of revolution (19) given by rotation of surface of revolution around axis ξ by angle θ, $\theta \neq 0$, $\theta \neq \pi$, see fig. 6, to obtain skew position of axes. Transformation matrix $\widetilde{\mathbf{G}}$ of this rotation is

$$\widetilde{\mathbf{G}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the solid generated by rotary motion of surface of revolution in this case is than given by

$\mathbf{B}(u, v, t) = \mathbf{S}(u, v) \cdot \widetilde{\mathbf{G}} \cdot \mathbf{G}(t), u \in [0,2\pi], v \in [0,1], t \in [0,2\pi].$

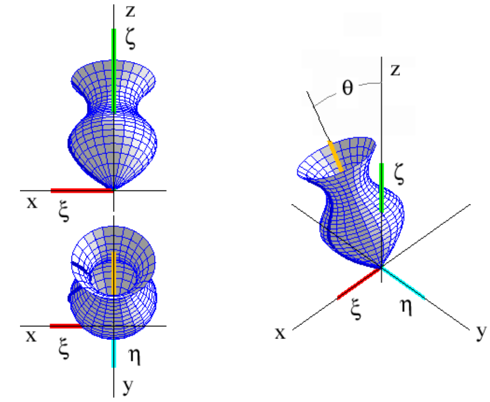

Fig. 7: Initial position of general surface of revolution (top, front and axonometric view)

The determinant in this case given by

$D(u, v) = x(u, v)x^v(u, v)z^u(u, v) - x(u, v)x^u(u, v)z^v(u, v) +$
$+rx^v(u, v)z^u(u, v) - rx^u(u, v)z^v(u, v) -$
$-y(u, v)y^u(u, v)z^v(u, v) + y(u, v)y^v(u, v)z^u(u, v)$

does not depend on t, see fig. 8, where the spatial graph of this surface is drawn. In this case, it is possible to find analytical solution of equation $D(u, v) = 0$ in form

$$u = u(v), u^* = u(v) + \pi, \tag{23}$$

i.e. the envelope surface is an explicit surface in the solid with two branches.

Graphical representation of solution (23) in parametric space is shown in fig. 9. Note that each branche of envelope surface in parametric space is a ruled surface (t-parametric curves of this surface are straight lines), i.e. the characteristic curve has non-variable shape.

Neither the analytical expression of the solution (23) nor envelope surface is given here because it is too large.Envelope surface in 3D space is drawn in fig. 10.

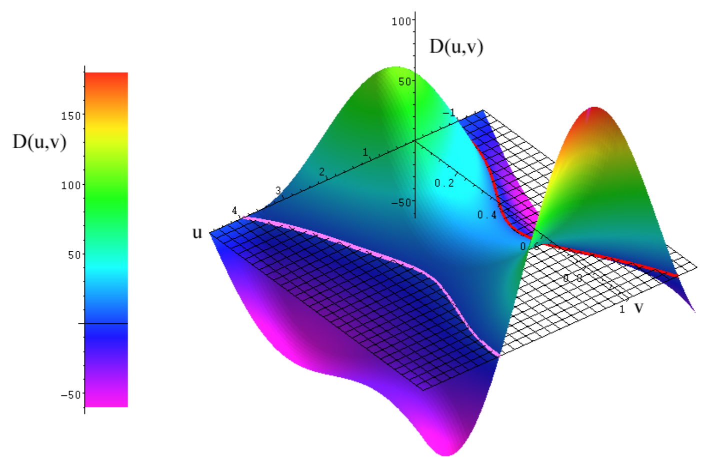

Fig. 8: Graphical solution of equation $D(u, v) = 0$for envelope surface generated by rotary motion of general surface of revolution when axis of rotary motion and axis of the generating surface are skew

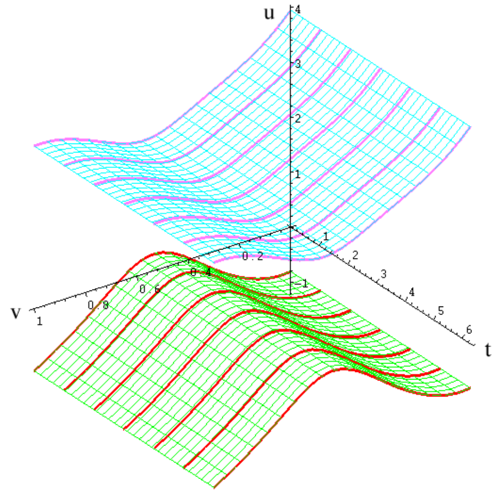

Fig. 9: Envelope surface (with several characteristic curves) generated by rotary motion of general surface of revolution when axis of rotary motion and axis of the generating surface are skew (mapping in parametric space (u, v, t) of the generated solid)

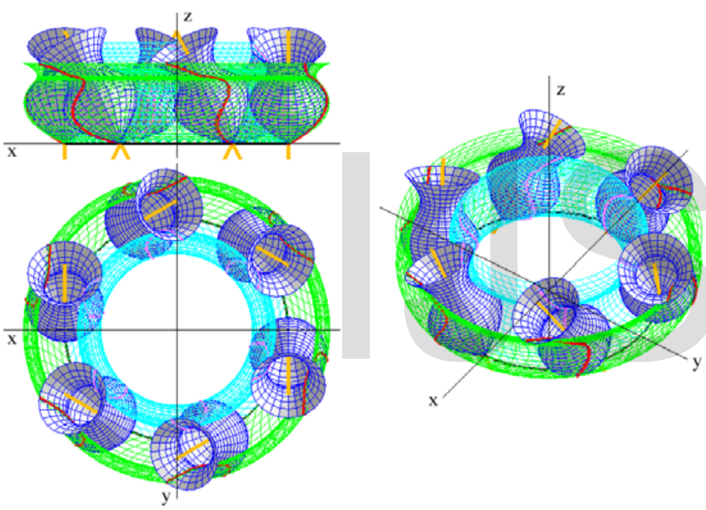

Fig. 10: Envelope surface generated by rotary motion of general surface of revolution when axis of rotary motion and axis of the generating surface are skew (top, front and axonometric view)

5.3Envelope Surface Is Explicit Surface of the Solid, Characteristic Curve Has Variable Shape

To give an example of envelope surface with variable shape of characteristic curve, the envelope surface generated by general motion of surface of revolution (19) along spatial freeform trajectory given by

$\mathbf{T}(t) = (x(t), y(t), z(t), 1), t \in [0,1]$,
$x(t) = -7t^3 + 6t^2 - 24t + 25$,
$y(t) = -78t^3 + 126t^2 - 54t$,
$z(t) = 30t^3 - 30t^2$

is considered, here. The transformation matrix of general curvilinear motion in this case is given by

$$\mathbf{G}(t) = \begin{pmatrix} \frac{c}{ab} & \frac{d}{ab} & \frac{e}{ab} & 0 \\ \frac{-7t^2+4t-8}{b} & \frac{6(-13t^2+14t-3)}{b} & \frac{10t(3t-2)}{b} & 0 \\ \frac{30(8t^2-9t+3)}{a} & \frac{5(t^2+24t-8)}{a} & \frac{3(23t^2+83t-50)}{a} & 0 \\ x(t) & y(t) & z(t) & 1 \end{pmatrix},$$

where

$a = \sqrt{62386t^4 - 94038t^3 + 17140t^2 - 132900t + 32200}$,
$b = \sqrt{7033t^4 - 14360t^3 + 10392t^2 - 3099t + 388}$
$c = 2(-2766t^4 - 856t^3 + 17487t^2 + 264t - 400)$,
$d = 3(2561t^4 - 3811t^3 + 2202t^2 + 264t - 400)$,
$e = 5(3737t^4 - 8408t^3 + 6948t^2 + 2708t - 388)$.

Neither the concrete analytical representation of the determinant (8) nor the solution of equation $D(u, v, t) = 0$ is presented here because it is too large. The determinant $D(u, v, t)$ is function of all three variables. Therefore, the graphical analysis along with the animation sequence is suitable to set up to solve this problem. An example of spatial graph for $t = 0.5$ of surface $D(u, v, 0.5)$ and plane $z = 0$ is drawn in fig. 11.

There are two intersection curves representing graphs of functions of v variable. In this case, it is possible to find solution of equation $D(u, v, t) = 0$ in explicit form

$$u = u(v, t), u^* = u(v, t) + \pi. \tag{24}$$

Variable shape of the two intersection curves is depicted in fig. 12. Top views of spatial graphs of surface $D(u, v, \gamma_i)$ and plane $z = 0$ for $t = 0, 0.1, \cdots, 1$ is drawn in several animation slides.

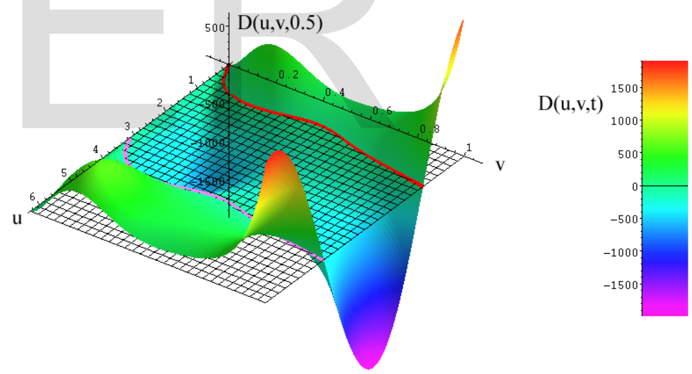

Fig. 11: Graphical solution of equation $D(u, v, t) = 0$ for envelope surface generated by general motion of general surface of revolution at $t = 0.5$

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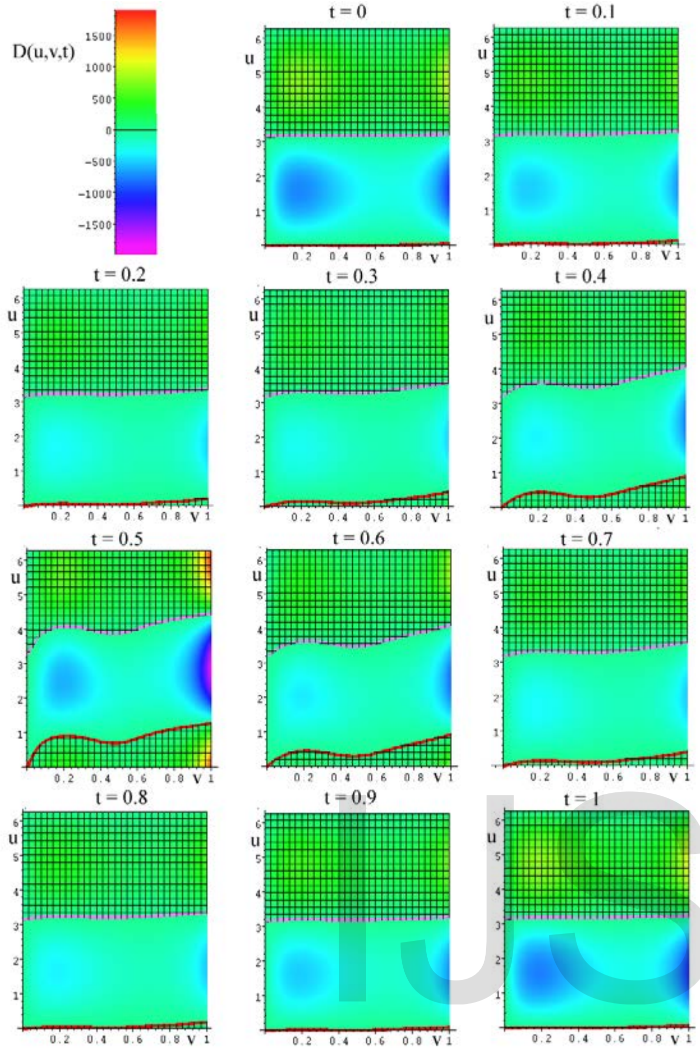

Fig. 12: Graphical solution of equation $D(u,v,t) = 0$ for envelope surface generated by general motion of general surface of revolution (top views of animation slides)

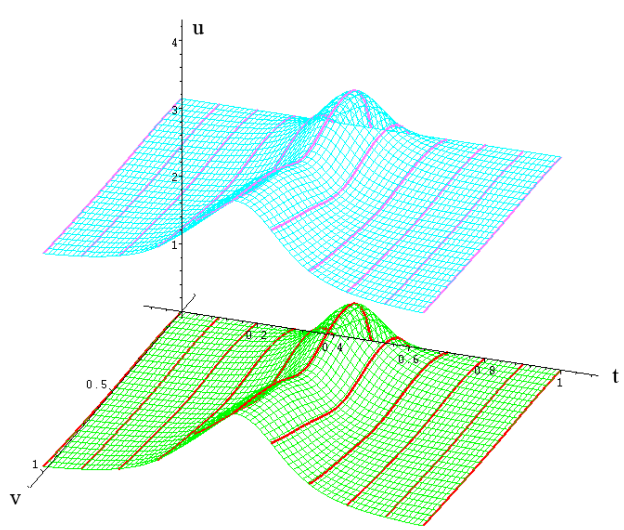

Fig. 13: Envelope surface with several characteristic curves) generated by general motion of general surface of revolution (mapping in parametric space (u, v, t) of the generated solid)

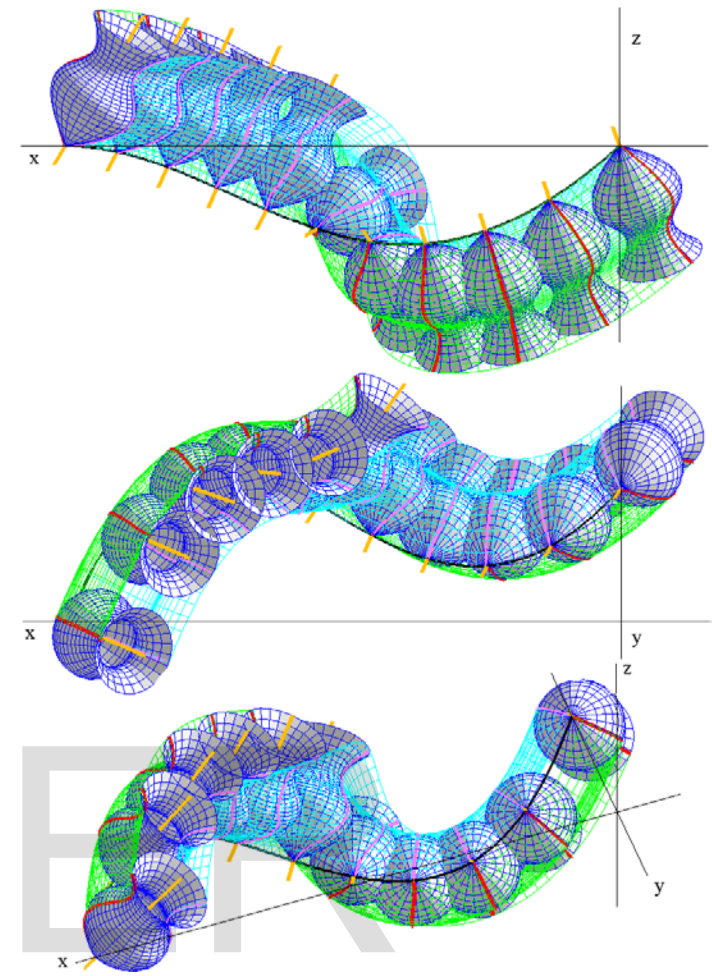

Fig. 14: Envelope surface generated by general motion of general surface of revolution (top, front and axonometric view)

Note that in case the explicit solution is impossible to find easily, these graphs can be used to find (u, v) coordinates of sufficient number of points along intersection curve. These points can be consequently fitted by suitable approximation or interpolation curve representing mathematical model of characteristic curve in parametric space (u, v, t) of the solid (4).

Graphical representation of solution (24) in parametric space is shown in fig. 13. Branches of envelope surface in parametric space are not ruled surface (t-parametric curves of this suface are not straight lines). It follows that the characteristic curve has variable shape.

In 3D space the shape of both parts of envelope surface as well as the shape of both branches of characteristic curve is not identical, see fig. 14, where the envelope surface generated by general motion of general surface of revolution is drawn.

6. Conclusion

A modified DG/K (Differential Geometry/Kinematics) approach to envelope surface solving is presented in this paper. A complex method has been developed to find the analytical representation of envelopesurface generated by continuous univariate motion of generating surface. The enter dataof this method is type of motion, analytic representation of generating surface and analyticrepresentation of trajectory in homogeneous coordinates of extended Euclidean space. Based on mapping of envelope surface in parametric space of the solid generated by continuous motion of general surface of revolution, it is possible to recognize the expected form of analytical representation of envelope surface and choose appropriate approach to find it. The obtained results can be used in mechanical engineering applications of envelope surfaces, especially in 5-axis flank milling of freeform surfaces.

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